

NEW HERMITE-HADAMARD AND SIMPSON TYPE INEQUALITIES FOR HARMONICALLY (s, m) -CONVEX FUNCTOINS IN SECOND SENSE

IMRAN ABBAS BALOCH, İMDAT İSCAN

ABSTRACT. In [1], authors introduced the concept of harmonically (s, m) -convex functions in second sense which unifies different type of convexities and is more general notion of Harmonic convexity. In this paper, authors obtain new estimates on generalization of Hermite-Hadamard and Simpson type inequalities for this larger class of functions.

1. Introduction

Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I and $a, b \in I$ with $a < b$, then following double inequalities hold

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The inequality (1.1) is known in the literature as Hermite-Hadamard integral inequality.

Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty$, then the following inequality holds

$$(1.2) \quad \left| \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4.$$

The inequality (1.2) is known in the literature as Simpson inequality. In recent years, many authors have studied errors estimates for Hermite-Hadamard and Simpson inequalities; for refinements, counterparts, generalization see [2,4,6,7,9,10].

In [1], authors introduced the concept of harmonically (s, m) -convex functions as follow

Definition 1.1. The function $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ is said to be harmonically (s, m) -convex in second sense, where $s \in (0, 1]$ and $m \in (0, 1]$ if

$$f\left(\frac{mxy}{mty + (1-t)x}\right) = f\left(\left(\frac{t}{x} + \frac{1-t}{my}\right)^{-1}\right) \leq t^s f(x) + m(1-t)^s f(y)$$

$\forall x, y \in I$ and $t \in [0, 1]$.

Remark 1.2. Note that for $s = 1$, harmonic (s, m) -convexity reduces to harmonic m -convexity and for $m = 1$, harmonic (s, m) -convexity reduces to harmonic s -convexity in second sense (see [5]) and for $s, m = 1$, harmonically (s, m) -convexity reduces to ordinary harmonic convexity (see [4]).

2. SOME BASIC PROPERTIES

In this section, we explore some basic results associated with harmonically (s, m) -convex functions in second sense.

Proposition 2.1. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a function

a) if f is (s, m) -convex function in second sense and non-decreasing, then f is harmonically (s, m) -convex function in second sense.

2010 *Mathematics Subject Classification.* Primary: 26D15. Secondary: 26A51.

Key words and phrases. Harmonically (s, m) -convex function, Hermite-Hadamard type inequalities, Simpson type inequalities.

b) if f is harmonically (s, m) -convex function in second sense and non-increasing, then f is (s, m) -convex function in second sense.

Remark 2.2. According to proposition 2.1, every non-decreasing (s, m) -convex function in second sense is also harmonically (s, m) -convex function in second sense.

Example 2.3. (see[2]) Let $0 < s < 1$ and $a, b, c \in \mathbb{R}$, then function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} a, & x = 0 \\ bx^s + c, & x > 0 \end{cases}$$

is non-decreasing s -convex function in second sense for $b \geq 0$ and $0 \leq c \leq a$. Hence, by proposition 2.1, f is harmonically $(s, 1)$ -convex function.

Proposition 2.4. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a harmonically (s, m) -convex in second sense, where $s, m \in (0, 1]$ and let a, b be nonnegative real numbers with $a < b$. Then for any $x \in [a, b]$, there is $t \in [0, 1]$ such that

$$f\left(\frac{ab}{a+b-x}\right) \leq t^s[f(a) + f(b)] + m(1-t)^s\left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right] - f\left(\frac{ab}{x}\right).$$

Proof. Since, any $x \in [a, b]$ can be represented as $x = ta + (1-t)b$, $t \in [0, 1]$, then

$$\begin{aligned} f\left(\frac{ab}{a+b-x}\right) &= f\left(\frac{ab}{a+b-ta-(1-t)b}\right) \\ &= f\left(\frac{ma(\frac{b}{m})}{mt(\frac{b}{m}) + (1-t)a}\right) \\ &\leq t^s f(a) + m(1-t)^s f\left(\frac{b}{m}\right) \\ &= t^s f(a) + m(1-t)^s f\left(\frac{b}{m}\right) + t^s f(b) - t^s f(b) + m(1-t)^s f\left(\frac{a}{m}\right) - m(1-t)^s f\left(\frac{a}{m}\right) \\ &= t^s[f(a) + f(b)] + m(1-t)^s\left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right] - f\left(\frac{ab}{ta + (1-t)b}\right) \\ &= t^s[f(a) + f(b)] + m(1-t)^s\left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right] - f\left(\frac{ab}{x}\right). \end{aligned}$$

□

Proposition 2.5. Let $f_i : (0, \infty) \rightarrow \mathbb{R}$, $i = 1, \dots, n$ are harmonically (s, m) -convex in second sense, where $s, m \in (0, 1]$, then function given by $f := \max_{i=1, \dots, n} \{f_i\}$ is also harmonically (s, m) -convex in second sense.

Proposition 2.6. Let $f_n : (0, \infty) \rightarrow \mathbb{R}$ be a sequence of harmonically (s, m) -convex in second sense, where $s, m \in (0, 1]$ and $f_n(x) \rightarrow f(x)$, then function f is also harmonically (s, m) -convex in second sense.

Proposition 2.7. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a harmonically (s_1, m) -convex in second sense and let $g : (0, \infty) \rightarrow \mathbb{R}$ be a harmonically (s_2, m) -convex in second sense, where $s_1, s_2, m \in (0, 1]$. Then $f + g$ is harmonically (s, m) -convex in second sense, where $s = \min\{s_1, s_2\}$.

Proposition 2.8. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a harmonically (s, m) -convex in second sense, where $s, m \in (0, 1]$. If $\lambda > 0$, then λf is harmonically (s, m) -convex in second sense.

Proposition 2.9. Let $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$ be a harmonically m -convex in second sense with $m \in (0, 1]$, and $g : I \subseteq f([0, b]) \rightarrow \mathbb{R}$ be nondecreasing and (s, m) -convex function in second sense on I for some fixed s , then $g \circ f$ is harmonically (s, m) -convex in second sense on $[0, b]$.

Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , throughout this article we will assume that

$$I_f(\lambda, \mu, a, b) = (\lambda - \mu)f\left(\frac{a+b}{2}\right) + (1 - \lambda)f(a) + \mu f(b) - \frac{2ab}{b-a} \int_a^b \frac{f(u)}{u^2} du,$$

where $a, b \in I$ with $a < b$ and $\lambda, \mu \in \mathbb{R}$.

In [7], I.İşcan et al established the following equality

Lemma 2.10. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. Then for all $\lambda, \mu \in \mathbb{R}$, we have*

$$I_f(\lambda, \mu, a, b) = ab(b-a) \left\{ \int_0^{\frac{1}{2}} \frac{\mu - t}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt + \int_{\frac{1}{2}}^1 \frac{\lambda - t}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt \right\},$$

where $A_t = tb + (1-t)a$

In this paper, we establish more general form of Hermite-hadamard and Simpson type inequalities by using Lemma 2.10 for harmonically (s, m) -convex functions.

3. MAIN RESULTS

Now, we present our main results which are more general in the following section. The Beta function, the Gamma function and the integral form of the hypergeometric function are defined as follows to be used in the sequel of paper

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt, \quad \alpha, \beta > 0$$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \quad \alpha > 0$$

and

$${}_2F_1(\alpha, \beta; \gamma, z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-zt)^{-\alpha} dt, \quad \gamma > \beta > 0, |z| < 1$$

Theorem 3.1. *Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, \frac{b}{m} \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically (s, m) -convex on $[a, \frac{b}{m}]$ for some fixed $q \geq 1$ and $0 \leq \mu \leq \frac{1}{2} \leq \lambda \leq 1$, then following inequality holds*

$$\begin{aligned} \left| I_f(\lambda, \mu, a, b) \right| &\leq ab(b-a) \left\{ \mathcal{B}_1^{1-\frac{1}{q}}(\mu) \left(|f'(a)|^q \mathcal{B}_2(\mu, q, a, b) + m |f'(\frac{b}{m})|^q \mathcal{B}_3(\mu, q, a, b) \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \mathcal{B}_4^{1-\frac{1}{q}}(\lambda) \left(|f'(a)|^q \mathcal{B}_5(\lambda, q, a, b) + m |f'(\frac{b}{m})|^q \mathcal{B}_6(\lambda, q, a, b) \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where

$$\mu^2 - \frac{\mu}{2} + \frac{1}{8} := \mathcal{B}_1(\mu)$$

,

$$\lambda^2 - \frac{3\lambda}{2} + \frac{5}{8} := \mathcal{B}_4(\lambda),$$

$$\mathcal{B}_2(\mu, q, a, b) = \begin{cases} \frac{2^{2q-s-2}\beta(1,s+2)}{(a+b)^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{2a}{b+a}), & \mu = 0 \\ \frac{2\mu^{s+2}\beta(2,s+1)}{[\mu b + (1-\mu)a]^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{a}{\mu b + (1-\mu)a}) - \frac{\mu^{2^{2q-s-2}\beta(1,s+1)}}{(b+a)^{2q}} \cdot {}_2F_1(2q, 1, s+2, 1 - \frac{2a}{b+a}) \\ + \frac{2^{2q-s-2}\beta(1,s+2)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{2a}{b+a}), & 0 < \mu < \frac{1}{2} \\ \frac{2^{2q-s-2}\beta(2,s+1)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{2a}{b+a}), & \mu = \frac{1}{2} \end{cases}$$

$$\begin{aligned}
\mathcal{B}_3(\mu, q, a, b) &= \begin{cases} \frac{\beta(s+1, s+3)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{a}{b}) - \frac{\beta(s+1, 1)}{2^{s+2}b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{b+a}{2b}) \\ - \frac{\beta(s+1, 2)}{2^{s+2}b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{b+a}{2b}), & \mu = 0 \\ \\ \frac{\mu\beta(s+1, 1)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{a}{b}) - \frac{\beta(s+1, 2)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{a}{b}) \\ + 2 \frac{(1-\mu)^{s+2}\beta(s+1, 2)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, (1-\mu)(1 - \frac{a}{b})) \\ + (\mu - 1) \frac{\beta(s+1, 1)}{2^{s+2}b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{b+a}{2b}) \\ + \frac{\beta(s+1, 2)}{2^{s+2}b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{b+a}{2b}), & 0 < \mu < \frac{1}{2} \\ \\ \frac{\beta(s+1, 1)}{2b^{2q}} \cdot {}_2F_1(2q, s+2, s+3, 1 - \frac{a}{b}) - \frac{\beta(s+1, 2)}{2b^{2q}} \cdot {}_2F_1(2q, s+2, s+3, 1 - \frac{a}{b}) \\ \frac{\beta(s+1, 2)}{2^{s+2}b^{2q}} \cdot {}_2F_1(2q, s+2, s+3, 1 - \frac{b+a}{2b}), & \mu = \frac{1}{2} \end{cases} \\
\mathcal{B}_5(\mu, q, a, b) &= \begin{cases} \frac{\beta(1, s+2)}{b^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{a}{b}) - \frac{2^{2q-s-2}\beta(1, s+2)}{(a+b)^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{2a}{b+a}), & \lambda = 0 \\ \\ \frac{2\lambda^{s+2}\beta(2, s+1)}{[\lambda b + (1-\lambda)a]^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{a}{\lambda b + (1-\lambda)a}) - \frac{\lambda 2^{2q-s-2}\beta(1, s+1)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 1, s+2, 1 - \frac{2a}{b+a}) \\ + \frac{2^{2q-s-2}\beta(1, s+2)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{2a}{b+a}) + \frac{\beta(1, s+2)}{b^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{a}{b}) \\ - \frac{\lambda\beta(1, s+1)}{b^{2q}} \cdot {}_2F_1(2q, 1, s+2, 1 - \frac{a}{b}), & 0 < \lambda < \frac{1}{2} \\ \\ \frac{\beta(1, s+2)}{2b^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{a}{b}) + \frac{2^{2q-s-2}\beta(2, s+1)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{2a}{b+a}) \\ - \frac{\beta(2, s+1)}{2b^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{a}{b}), & \lambda = \frac{1}{2} \end{cases} \\
\mathcal{B}_6(\mu, q, a, b) &= \begin{cases} \frac{\beta(s+1, 1)}{2^{s+2}b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{b+a}{2b}) - \frac{\beta(s+1, 2)}{2^{s+1}b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{b+a}{2b}), & \lambda = 0 \\ \\ \frac{2\lambda\beta(s+1, 1)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{a}{b}) + (\lambda - 1) \frac{\beta(1, s+1)}{2^{s+1}b^{2q}} \cdot {}_2F_1(2q, 1, s+2, 1 - \frac{b+a}{2b}) \\ + 2 \frac{(1-\lambda)^{s+2}\beta(s+1, 2)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, (1-\lambda)(1 - \frac{a}{b})) \\ + \frac{\beta(2, s+1)}{2^{s+1}b^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{b+a}{2b}), & 0 < \lambda < \frac{1}{2} \\ \\ \frac{\beta(s+1, 2)}{2^{s+2}b^{2q}} \cdot {}_2F_1(2q, s+2, s+3, 1 - \frac{b+a}{2b}), & \lambda = \frac{1}{2} \end{cases}
\end{aligned}$$

Proof. using Lemma 2.10, Hölder's inequality and harmonically (s, m) -convexity in second sense of $|f'|^q$, we get

$$\begin{aligned}
\left| I_f(\lambda, \mu, a, b) \right| &\leq ab(b-a) \left\{ \left(\int_0^{\frac{1}{2}} |\mu - t| dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \frac{|\mu - t|}{A_t^{2q}} |f' \left(\frac{ab}{A_t} \right)|^q \right)^{\frac{1}{q}} \right. \\
&\quad + \left. \left(\int_{\frac{1}{2}}^1 |\lambda - t| dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 \frac{|\lambda - t|}{A_t^{2q}} |f' \left(\frac{ab}{A_t} \right)|^q \right)^{\frac{1}{q}} \right\} \\
&\leq ab(b-a) \left\{ \left(\int_0^{\frac{1}{2}} |\mu - t| dt \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left(|f'(a)|^q \int_0^{\frac{1}{2}} \frac{|\mu - t| t^s}{A_t^{2q}} dt + m |f'(\frac{b}{m})|^q \int_0^{\frac{1}{2}} \frac{|\mu - t| (1-t)^s}{A_t^{2q}} \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\frac{1}{2}}^1 |\lambda - t| dt \right)^{1-\frac{1}{q}} \\
&\quad \times \left(|f'(a)|^q \int_{\frac{1}{2}}^1 \frac{|\lambda - t| t^s}{A_t^{2q}} dt + m |f'(\frac{b}{m})|^q \int_{\frac{1}{2}}^1 \frac{|\lambda - t| (1-t)^s}{A_t^{2q}} \right)^{\frac{1}{q}},
\end{aligned}$$

where, by calculations we find that

$$\begin{aligned}
& \int_0^{\frac{1}{2}} |\mu - t| dt = \mu^2 - \frac{\mu}{2} + \frac{1}{8}, \quad \int_{\frac{1}{2}}^1 |\lambda - t| dt = \lambda^2 - \frac{3\lambda}{2} + \frac{5}{8}, \\
& \int_0^{\frac{1}{2}} \frac{|\mu - t| t^s}{A_t^{2q}} dt = \begin{cases} \frac{2^{2q-s-2} \beta(1, s+2)}{(a+b)^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{2a}{b+a}), & \mu = 0 \\ \frac{2\mu^{s+2} \beta(2, s+1)}{[\mu b + (1-\mu)a]^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{a}{\mu b + (1-\mu)a}) - \frac{\mu 2^{2q-s-2} \beta(1, s+1)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 1, s+2, 1 - \frac{2a}{b+a}) \\ + \frac{\mu 2^{2q-s-2} \beta(1, s+2)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{2a}{b+a}), & 0 < \mu < \frac{1}{2} \\ \frac{2^{2q-s-2} \beta(2, s+1)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{2a}{b+a}), & \mu = \frac{1}{2} \end{cases} \\
& \int_0^{\frac{1}{2}} \frac{|\mu - t| (1-t)^s}{A_t^{2q}} dt = \begin{cases} \frac{\beta(s+1, s+3)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{a}{b}) - \frac{\beta(s+1, 1)}{2^{s+2} b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{b+a}{2b}) \\ - \frac{\beta(s+1, 2)}{2^{s+2} b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{b+a}{2b}), & \mu = 0 \\ \frac{\mu \beta(s+1, 1)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{a}{b}) - \frac{\beta(s+1, 2)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{a}{b}) \\ + 2 \frac{(1-\mu)^{s+2} \beta(s+1, 2)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, (1-\mu)(1 - \frac{a}{b})) \\ + (\mu - 1) \frac{\beta(s+1, 1)}{2^{s+2} b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{b+a}{2b}) \\ + \frac{\beta(s+1, 2)}{2^{s+2} b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{b+a}{2b}), & 0 < \mu < \frac{1}{2} \\ \frac{\beta(s+1, 1)}{2b^{2q}} \cdot {}_2F_1(2q, s+2, s+3, 1 - \frac{a}{b}) - \frac{\beta(s+1, 2)}{2b^{2q}} \cdot {}_2F_1(2q, s+2, s+3, 1 - \frac{a}{b}) \\ \frac{\beta(s+1, 2)}{2^{s+2} b^{2q}} \cdot {}_2F_1(2q, s+2, s+3, 1 - \frac{b+a}{2b}), & \mu = \frac{1}{2} \end{cases} \\
& \int_{\frac{1}{2}}^1 \frac{|\lambda - t| t^s}{A_t^{2q}} dt = \begin{cases} \frac{\beta(1, s+2)}{b^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{a}{b}) - \frac{2^{2q-s-2} \beta(1, s+2)}{(a+b)^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{2a}{b+a}), & \lambda = 0 \\ \frac{2\lambda^{s+2} \beta(2, s+1)}{[\lambda b + (1-\lambda)a]^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{a}{\lambda b + (1-\lambda)a}) - \frac{\lambda 2^{2q-s-2} \beta(1, s+1)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 1, s+2, 1 - \frac{2a}{b+a}) \\ + \frac{2^{2q-s-2} \beta(1, s+2)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{2a}{b+a}) + \frac{\beta(1, s+2)}{b^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{a}{b}) \\ - \frac{\lambda \beta(1, s+1)}{b^{2q}} \cdot {}_2F_1(2q, 1, s+2, 1 - \frac{a}{b}), & 0 < \lambda < \frac{1}{2} \\ \frac{\beta(1, s+2)}{2b^{2q}} \cdot {}_2F_1(2q, 1, s+3, 1 - \frac{a}{b}) + \frac{2^{2q-s-2} \beta(2, s+1)}{(b+a)^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{2a}{b+a}) \\ - \frac{\beta(2, s+1)}{2b^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{a}{b}), & \lambda = \frac{1}{2} \end{cases} \\
& \int_{\frac{1}{2}}^1 \frac{|\lambda - t| (1-t)^s}{A_t^{2q}} dt = \begin{cases} \frac{\beta(s+1, 1)}{2^{s+2} b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{b+a}{2b}) - \frac{\beta(s+1, 2)}{2^{s+2} b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, 1 - \frac{b+a}{2b}), & \lambda = 0 \\ \frac{2\lambda \beta(s+1, 1)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+2, 1 - \frac{a}{b}) + (\lambda - 1) \frac{\beta(1, s+1)}{2^{s+1} b^{2q}} \cdot {}_2F_1(2q, 1, s+2, 1 - \frac{b+a}{2b}) \\ + 2 \frac{(1-\lambda)^{s+2} \beta(s+1, 2)}{b^{2q}} \cdot {}_2F_1(2q, s+1, s+3, (1-\lambda)(1 - \frac{a}{b})) \\ + \frac{\beta(2, s+1)}{2^{s+1} b^{2q}} \cdot {}_2F_1(2q, 2, s+3, 1 - \frac{b+a}{2b}), & 0 < \lambda < \frac{1}{2} \\ \frac{\beta(s+1, 2)}{2^{s+2} b^{2q}} \cdot {}_2F_1(2q, s+2, s+3, 1 - \frac{b+a}{2b}), & \lambda = \frac{1}{2} \end{cases}
\end{aligned}$$

which completes the proof. \square

Corollary 3.2. *Under the assumption of Theorem 3.1 with $\lambda = \mu = \frac{1}{2}$, the inequality (3.1) reduced to the following inequality*

$$\begin{aligned}
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| & \leq ab(b-a) \left(\frac{1}{8} \right)^{1-\frac{1}{q}} \left\{ \left(|f'(a)|^q \mathcal{B}_2\left(\frac{1}{2}, q, a, b\right) + m |f'\left(\frac{b}{m}\right)|^q \mathcal{B}_3\left(\frac{1}{2}, q, a, b\right) \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(|f'(a)|^q \mathcal{B}_5\left(\frac{1}{2}, q, a, b\right) + m |f'\left(\frac{b}{m}\right)|^q \mathcal{B}_6\left(\frac{1}{2}, q, a, b\right) \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

Corollary 3.3. *Under the assumption of Theorem 3.1 with $\lambda = 1$ and $\mu = 0$, the inequality (3.1) reduced to the following inequality*

$$\left| f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq ab(b-a) \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \left\{ \left(|f'(a)|^q \mathcal{B}_2(0, q, a, b) + m |f'\left(\frac{b}{m}\right)|^q \mathcal{B}_3(0, q, a, b) \right)^{\frac{1}{q}} \right. \\ \left. + \left(|f'(a)|^q \mathcal{B}_5(1, q, a, b) + m |f'\left(\frac{b}{m}\right)|^q \mathcal{B}_6(1, q, a, b) \right)^{\frac{1}{q}} \right\}$$

Corollary 3.4. *Under the assumption of Theorem 3.1 with $\lambda = \frac{5}{6}$ and $\mu = \frac{1}{6}$, the inequality (3.1) reduced to the following inequality*

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{2ab}{a+b}\right) \right] - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \\ \leq ab(b-a) \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left\{ \left(|f'(a)|^q \mathcal{B}_2\left(\frac{1}{6}, q, a, b\right) + m |f'\left(\frac{b}{m}\right)|^q \mathcal{B}_3\left(\frac{1}{6}, q, a, b\right) \right)^{\frac{1}{q}} \right. \\ \left. + \left(|f'(a)|^q \mathcal{B}_5\left(\frac{5}{6}, q, a, b\right) + m |f'\left(\frac{b}{m}\right)|^q \mathcal{B}_6\left(\frac{5}{6}, q, a, b\right) \right)^{\frac{1}{q}} \right\}$$

Theorem 3.5. *Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, \frac{b}{m} \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically (s, m) -convex on $[a, \frac{b}{m}]$ for some fixed $q > 1$ and $0 \leq \mu \leq \frac{1}{2} \leq \lambda \leq 1$, then following inequality holds*

$$\left| I_f(\lambda, \mu, a, b) \right| \leq ab(b-a) \left\{ \mathcal{B}_7^{\frac{1}{p}}(\mu) \left(|f'(a)|^q \mathcal{B}_8(\mu, q, a, b) + m |f'\left(\frac{b}{m}\right)|^q \mathcal{B}_9(\mu, q, a, b) \right)^{\frac{1}{q}} \right. \\ \left. + \mathcal{B}_{10}^{\frac{1}{p}}(\lambda) \left(|f'(a)|^q \mathcal{B}_{11}(\lambda, q, a, b) + m |f'\left(\frac{b}{m}\right)|^q \mathcal{B}_{12}(\lambda, q, a, b) \right)^{\frac{1}{q}} \right\}$$

where

$$\mathcal{B}_7(\mu) = \frac{1}{p+1} \left[\mu^{p+1} + \left(\frac{1}{2} - \mu \right)^{p+1} \right],$$

$$\mathcal{B}_{10}(\lambda) = \frac{1}{p+1} \left[\left(\lambda - \frac{1}{2} \right)^{p+1} + \left(1 - \lambda \right)^{p+1} \right],$$

$$\mathcal{B}_8(\mu, q, a, b) = \frac{2^{2q-s-2} \beta(1, s+1)}{(b+a)^{2q}} {}_2F_1(2q, 1, s+2, 1 - \frac{2a}{b+a})$$

$$\mathcal{B}_9(\mu, q, a, b) = \frac{\beta(s+1, 1)}{b^{2q}} {}_2F_1(2q, s+1, s+2, 1 - \frac{a}{b}) - \frac{\beta(s+1, 2)}{2^{s+1} b^{2q}} {}_2F_1(2q, s+1, s+2, 1 - \frac{a+b}{2b})$$

$$\mathcal{B}_{11}(\lambda, q, a, b) = \frac{\beta(1, s+1)}{b^{2q}} {}_2F_1(2q, 1, s+2, 1 - \frac{a}{b}) - \frac{2^{2q-s-1} \beta(1, s+1)}{(b+a)^{2q}} {}_2F_1(2q, 1, s+2, 1 - \frac{2a}{b+a})$$

$$\mathcal{B}_{12}(\mu, q, a, b) = \frac{\beta(s+1, 2)}{2^{s+1} b^{2q}} {}_2F_1(2q, s+1, s+3, 1 - \frac{a+b}{2b})$$

Proof. using Lemma 2.10, Hölder's inequality and harmonically (s, m) -convexity in second sense of $|f'|^q$, we get

$$\begin{aligned}
\left| I_f(\lambda, \mu, a, b) \right| &\leq ab(b-a) \left\{ \left(\int_0^{\frac{1}{2}} |\mu - t|^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \frac{1}{A_t^{2q}} |f' \left(\frac{ab}{A_t} \right)|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad + \left. \left(\int_{\frac{1}{2}}^1 |\lambda - t|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \frac{1}{A_t^{2q}} |f' \left(\frac{ab}{A_t} \right)|^q dt \right)^{\frac{1}{q}} \right\} \\
&\leq ab(b-a) \left\{ \left(\int_0^{\frac{1}{2}} |\mu - t|^p dt \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(|f'(a)|^q \int_0^{\frac{1}{2}} \frac{t^s}{A_t^{2q}} dt + m |f' \left(\frac{b}{m} \right)|^q \int_0^{\frac{1}{2}} \frac{(1-t)^s}{A_t^{2q}} dt \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\frac{1}{2}}^1 |\lambda - t|^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(|f'(a)|^q \int_{\frac{1}{2}}^1 \frac{t^s}{A_t^{2q}} dt + m |f' \left(\frac{b}{m} \right)|^q \int_{\frac{1}{2}}^1 \frac{(1-t)^s}{A_t^{2q}} dt \right)^{\frac{1}{q}},
\end{aligned}$$

where, by calculations we find that

$$\begin{aligned}
\left(\int_0^{\frac{1}{2}} |\mu - t|^p dt \right)^{\frac{1}{p}} &= \frac{1}{p+1} \left[\mu^{p+1} + \left(\frac{1}{2} - \mu \right)^{p+1} \right] \\
\left(\int_{\frac{1}{2}}^1 |\lambda - t|^p dt \right)^{\frac{1}{p}} &= \frac{1}{p+1} \left[\left(\lambda - \frac{1}{2} \right)^{p+1} + \left(1 - \lambda \right)^{p+1} \right] \\
\int_0^{\frac{1}{2}} \frac{t^s}{A_t^{2q}} dt &= \frac{2^{2q-s-2} \beta(1, s+1)}{(b+a)^{2q}} {}_2F_1(2q, 1, s+2, 1 - \frac{2a}{b+a}) \\
\int_0^{\frac{1}{2}} \frac{(1-t)^s}{A_t^{2q}} dt &= \frac{\beta(s+1, 1)}{b^{2q}} {}_2F_1(2q, s+1, s+2, 1 - \frac{a}{b}) - \frac{\beta(s+1, 2)}{2^{s+1} b^{2q}} {}_2F_1(2q, s+1, s+2, 1 - \frac{a+b}{2b}) \\
\int_{\frac{1}{2}}^1 \frac{t^s}{A_t^{2q}} dt &= \frac{\beta(1, s+1)}{b^{2q}} {}_2F_1(2q, 1, s+2, 1 - \frac{a}{b}) - \frac{2^{2q-s-1} \beta(1, s+1)}{(b+a)^{2q}} {}_2F_1(2q, 1, s+2, 1 - \frac{2a}{b+a}) \\
\int_{\frac{1}{2}}^1 \frac{(1-t)^s}{A_t^{2q}} dt &= \frac{\beta(s+1, 2)}{2^{s+1} b^{2q}} {}_2F_1(2q, s+1, s+3, 1 - \frac{a+b}{2b})
\end{aligned}$$

which completes the proof. \square

Corollary 3.6. *Under the assumption of Theorem 3.5 with $\lambda = \mu = \frac{1}{2}$, the inequality (3.1) reduced to the following inequality*

$$\begin{aligned}
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq ab(b-a) \left(\frac{1}{(p+1)2^{p+1}} \right)^{\frac{1}{p}} \left\{ \left(|f'(a)|^q \mathcal{B}_8(q, a, b) + m |f' \left(\frac{b}{m} \right)|^q \mathcal{B}_9(q, a, b) \right)^{\frac{1}{q}} \right. \\
&\quad + \left. \left(|f'(a)|^q \mathcal{B}_{11}(q, a, b) + m |f' \left(\frac{b}{m} \right)|^q \mathcal{B}_{12}(q, a, b) \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

Corollary 3.7. *Under the assumption of Theorem 3.5 with $\lambda = 1$ and $\mu = 0$, the inequality (3.1) reduced to the following inequality*

$$\begin{aligned}
\left| f \left(\frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| &\leq ab(b-a) \left(\frac{1}{(p+1)2^{p+1}} \right)^{\frac{1}{p}} \left\{ \left(|f'(a)|^q \mathcal{B}_8(q, a, b) + m |f' \left(\frac{b}{m} \right)|^q \mathcal{B}_9(q, a, b) \right)^{\frac{1}{q}} \right. \\
&\quad + \left. \left(|f'(a)|^q \mathcal{B}_{11}(q, a, b) + m |f' \left(\frac{b}{m} \right)|^q \mathcal{B}_{12}(q, a, b) \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

Corollary 3.8. *Under the assumption of Theorem 3.5 with $\lambda = \frac{5}{6}$ and $\mu = \frac{1}{6}$, the inequality (3.1) reduced to the following inequality*

$$\begin{aligned} & \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{2ab}{a+b}\right) \right] - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \\ & \leq ab(b-a) \left(\frac{(2^{p+1})}{(p+1)6^{p+1}} \right)^{\frac{1}{p}} \left\{ \left(|f'(a)|^q \mathcal{B}_2(q, a, b) + m |f'(\frac{b}{m})|^q \mathcal{B}_3(q, a, b) \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(|f'(a)|^q \mathcal{B}_5(q, a, b) + m |f'(\frac{b}{m})|^q \mathcal{B}_6(q, a, b) \right)^{\frac{1}{q}} \right\} \end{aligned}$$

REFERENCES

- [1] I. A. Baloch, I.İşcan, Some Ostrowski Type Inequalities For Harmonically (s, m) -convex functoins in Second Sense, International Journal of Analysis, Volume 2015, Article ID 672675, 9 pages.
<http://dx.doi.org/10.1155/2015/672675>
- [2] W.W. Breckner, "Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, Publ.Inst.Math.(Beograd), 23, (1978), 13-20.
- [3] F.Chen and S.Wu, "Some Hermite-Hadamard type inequalities for harmonically s -convex functions," The scientific World Journal, vol 2014, Article ID 279158, 7 pages, 2014.
- [4] İ.İşcan, "Hermite-Hadamard type inequalities for harmonically convex functions," Hacettepe Journal of Mathematics and statistics, vol 43 (6) (2014), 935-942.
- [5] İ.İşcan, "Ostrowski type inequalities for harmonically s -convex functions," Konuralp journal of Mathematics, 3(1) (2015), 63-74 .
- [6] İ.İşcan, "Hermite-Hadamard type inequalities for harmonically (α, m) convex functions," Hacettepe Journal of Mathematics and statistics. Accepted for publication "arXiv:1307.5402v2[math.CA]".
- [7] İ.İşcan, S. Numan and K. Bekar, "Hermite-Hadamard and Simpsontype inequalities for differentiable harmonically P-functions," British Journal of Mathematics and Compute Science, 4(14): 1908-1920, 2014.
- [8] J.Park, "New Ostrowski-Like type inequalities for differentiable (s, m) -convex mappings, International journal of pure and applied mathematics, vol.78 No.8 2012, 1077-1089.
- [9] E.Set, İ.İşcan, F.Zehir, On some new inequalities of Hermite-Hadamard type involving harmonically convex function via fractional integrals, "Konuralp journal of Mathematics, 3(1) (2015), 42-55 .
- [10] Özdemir ME, Yildiz C, New inequalities for Hermite-Hadamard and Simpson Type with applications. Tamkang Journal of Mathematics. 2013;44(2):209-216.

IMRAN ABBAS BALOCH, ABDUS SALAM SCHOOL OF MATHEMATICAL SCIENCES, GC UNIVERSITY, LAHORE, PAKISTAN
E-mail address: iabbasbaloch@gmail.com, iabbasbaloch@sms.edu.pk

İMDAT İSCAN, DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, GİRESUN UNIVERSITY, 28200, GİRESUN, TURKEY
E-mail address: imdat.iscan@giresun.edu.tr